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ON THE DETERMINATION OF MINIMAX FLOW CENTERS IN TREES

by

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ABSTRACT

In this paper we consider the following problem. Let a tree be given with supplies associated with its vertices. Then the objective is to find a minimax flow center v* which is defined as a vertex minimizing the largest edge flow which results from shipping all supplies to v*. We prove that in trees minimax flow centers coincide with medians and that they can thus be determined in linear time.

Key words : facility location; minimax flow problem; medians; location on trees.

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RÉSUMÉ

Étant donnē un arbre et des offres d'un produit aux sommets de cet arbre, nous considérons le problème de la détermination d'un centre de flot minimax sur cet arbre, c'est-à-dire d'un sommet tel que si toutes les quantités du produit prēsentes dans l'arbre sont envoyées à ce sommet, le plus grand flot résultant sur une arête soit minimisē. Nous démontrons que, dans les arbres, les centres de flot minimax coïncident avec les mēdianes, et qu'ils peuvent donc être déterminēs par un algorithme en temps linéaire.

Mots-clēs : localisation sur les arbres; problèmes de flot minimax; mēdianes; localisation de ressource.

Let a tree $T=(V,E)$ be given with a set of vertices $V = \{v_1,\ldots,v_n\}$ and a set of edges $E = \{e_1,\ldots,e_m\}$. With each vertex, a positive supply $s(v_i) > 0$, $v_i \in V$ is associated. A facility is to be located at a vertex of the tree, so that the supplies of all vertices are shipped to that facility. The objective is to locate the facility, so that the largest flow on any edge in the associated flow pattern is as small as possible. Such a vertex is called a minimax flow center (MMFC) of the tree.

Applications of minimax problems on graphs are found in a variety of situations, e.g. in the military context [2], in communication networks [5] and many others [1]. Minimax flow centers in complete graphs and in trees have been studied by Eiselt and Pederzoli [2] who have proposed an $O(n^2)$ algorithm for their determination in trees.

In this paper, we show that in trees minimax flow centers coincide with vertex-medians and that they can thus be determined in linear time ($O(n)$). We also state conditions relative to the unicity of MMFCs.

Let us first introduce some necessary notation. For any subtree $T'=(V',E')$ of T, the supply of that subtree is defined as $s(T') = \sum_{v_i \in V'} s(v_i)$. For simplicity, let $s(T)=S$. Given that the facility to which all supplies are shipped is located at v_i, then the associated flow is denoted by $f^i(e_k)$ for all $e_k \in E$ and the flow pattern is $f^i = \{f^i(e_k)\}$. The problem is then to find a vertex v^* and its objective value z^* so as to

$$\left.\begin{array}{l} \text{Minimize } z_i = \max\limits_{e_k \in E} \{f^i(e_k)\} \\ \\ \text{subject to the constraint that all supplies are shipped to } v_i \text{ .} \end{array}\right\} \quad (1)$$

We shall now prove a lemma which will lead to the fundamental result of the paper.

Suppose that the deletion of edge e_k from E divides T into two subtrees T_1^k and T_2^k. Then

Lemma 1: $z^* = \max_{e_k \in E} \left\{ \min \{ s(T_1^k); s(T_2^k) \} \right\} \leq S/2.$

Proof : Let v^* be an MMFC and let $f^* = \{ f^*(e_k) \}$ denote the unique flow pattern induced by locating the facility at v^*. By definition, we have

$$z^* = \max_{e_k \in E} \{ f^*(e_k) \} . \tag{2}$$

First we will prove that

$$f^*(e_k) = \min \{ s(T_1^k); s(T_2^k) \} \text{ for all } e_k \in E .$$

For that purpose, consider an arbitrary edge $e_k = (v_1^k, v_2^k)$, so that $v_i^k \in T_i^k$, i=1,2. Without loss of generality, let $v^* \in T_1^k$. As all supplies in T_2^k are shipped to v^* via e_k, we obtain

$$f^*(e_k) = s(T_2^k) . \tag{3}$$

If it can be shown that

$$s(T_2^k) \leq s(T_1^k) , \tag{4}$$

then (2), (3) and (4) imply the first part of the relation in lemma 1. In order to do so, suppose that $s(T_2^k) > s(T_1^k)$ and consider the flow pattern f' that would result from locating the facility at vertex v_2^k. Then the following relations hold:

$$f'(e_\ell) < s(T_1^k) \text{ for all } e_\ell \text{ in } T_1^k ,$$

$$f'(e_k) = s(T_1^k) ,$$

which, as $s(T_1^k) < s(T_2^k)$ is assumed, implies

$$f'(e_\ell) < s(T_2^k) \text{ for all } e_\ell \text{ in } T_1^k .$$

Furthermore, $f'(e_\ell) < s(T_2^k) - s(v_2^k)$ for all e_ℓ in T_2^k, which, with $s(v_2^k) > 0$, implies $f'(e_\ell) < s(T_2^k)$ for all e_ℓ in T_2^k.

Therefore, $f'(e_\ell) < s(T_2^k) < z^*$ for all $e_\ell \varepsilon E$ which contradicts that v^* is an MMFC.

Thus

$$f^*(e_k) = \min \{s(T_1^k); s(T_2^k)\} \quad \text{for all } e_k \varepsilon E . \tag{5}$$

Note that for any $e_k \varepsilon E$, $s(T_1^k) + s(T_2^k) = S$ which implies

$$\min \{s(T_1^k); s(T_2^k)\} < S/2 \quad \text{for all } e_k \varepsilon E . \tag{6}$$

From (2), (5) and (6) we may conclude that

$$z^* = \max_{e_k \varepsilon E} \left\{ \min \{s(T_1^k); s(T_2^k)\} \right\} < S/2 . \quad \blacksquare$$

Suppose now that vertex v_i has a degree of $|E_i|$, so that its removal from V along with all edges incident to it generates subtrees T_1^i, T_2^i, ..., $T_{|E_i|}^i$. Then we can state

Theorem 2: The vertex v_i is an MMFC if and only if it is a vertex-median of T.

Proof: From the theorem proven by Goldman and Witzgall [5], it can easily be inferred that v_i is a vertex-median of T if and only if

$$s(T_j^i) \leq S/2, \; j = 1,\ldots,|E_i| \; . \tag{7}$$

We shall now prove that v_i is an MMFC if and only if (7) is satisfied.

Let e_j^i, $j = 1,\ldots,|E_i|$ denote the unique edge connecting T_j^i to v_i. Locating the facility at v_i implies that

$$f^i(e_j^i) = s(T_j^i), \; j = 1,\ldots,|E_i| \; ,$$

and

$$f^i(e_\ell) < s(T_j^i), \text{ for all } e_\ell \in T_j^i, \; j = 1,\ldots,|E_i| \; ,$$

and thus

$$\max_{e_k \in E} \{f^i(e_k)\} = \max_j \{s(T_j^i)\} \; . \tag{8}$$

First we prove the necessity of condition (7). If (7) is not satisfied, then there exists exactly one index j such that $s(T_j^i) > S/2$ and thus $f^i(e_j^i) > S/2$. According to lemma 1, v_i cannot be an MMFC.

We now prove the sufficiency of (7) by contradiction. Let condition (7) be satisfied, but assume that v_i is not an MMFC. Then there must be a vertex $v' \neq v_i$ which is an MMFC. Without loss of generality, suppose that v' is in T_1^i. Then

$$f'(e_1^i) = S - s(T_1^i)$$

$$> S/2 \qquad \text{[by (7)]}$$

or simply $f'(e_1^i) > S/2$. Then

$$z' = \max_{e_k \in E} \{f'(e_k)\} \geq f'(e_1^i)$$

$$\geq S/2 \qquad \text{[by (9)]}$$

$$\geq s(T_j^i),\ j = 1,\ldots,|E_i| \qquad \text{[by (7)]}$$

$$\geq \max_{e_k \in E} \{f^i(e_k)\}. \qquad \text{[by (8)]}$$

Since by (1), $\max_{e_k \in E} \{f^i(e_k)\} = z_i$, $z' \geq z_i$ and thus v_i must be an MMFC. ∎

The next three corollaries follow directly from similar results regarding vertex-medians on trees (Goldman [4]).

Corollary 3: The vertex v_i is the unique MMFC if and only if

$$s(T_j^i) < S/2 \quad \text{for all } j = 1,\ldots,|E_i| \ . \qquad (10)$$

Corollary 4: Given that $s(v_i) > 0$, for all $v_i \in V$, a tree T admits exactly two MMFCs if and only if there exists an edge $e_k = (v_1^k, v_2^k)$, such that $s(T_1^k) = s(T_2^k) = S/2$. The two MMFCs are then v_1^k and v_2^k.

Corollary 5: The MMFCs of a tree can be determined in linear time (O(n) operations).

To determine the MMFCs of a tree, one can use Goldman's [4] algorithm. This procedure has to be adapted, if one wants to compute the value of the objective function z*, but the required modification is straightforward.

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